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# A spaceability result in the context of hypergroups

**Seyyed Mohammad Tabatabaie**

*Department of Mathematics, University of Qom, Qom, Iran*  
 sm.tabatabaie@qom.ac.ir

**Bentol Hoda Sadathoseyni**

*Department of Mathematics, University of Qom, Qom, Iran*  
 sb.sadathosseini@stu.qom.ac.ir

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**Abstract.** In this paper, by an elementary constructive technique, it is shown that  $L^r(\mathbb{Z}_+) - \bigcup_{q < r} L^q(\mathbb{Z}_+)$  is non-empty, where  $\mathbb{Z}_+$  is the dual of a compact countable hypergroup introduced by Dunkl and Ramirez. Also, we prove that for each  $r > 1$ ,  $L^r(\mathbb{Z}_+) - \bigcup_{q < r} L^q(\mathbb{Z}_+)$  is spaceable.

**Keywords:** locally compact hypergroup, spaceability,  $L^p$ -space

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## 1 Introduction and Notations

Suppose that  $X$  is a topological vector space. A subset  $S \subseteq X$  is called *spaceable* if  $S \cup \{0\}$  is large enough to contain a closed infinite dimensional subspace of  $X$ . This concept was first introduced in [6] and then the term *spaceable* was used in [1]. There have been several further works on this notion (for example see [4], [7] and [9]). It is well-known that for each  $0 < p$ , there are sequences in  $l_p$  not belonging to  $\bigcup_{q < p} l_q$ . Usually, this fact is proved by non-constructive techniques, although a constructive proof, depending on the Principle of Uniform Boundedness can be found in [8]. Actually, in [3] it is proved that for every  $p > 0$ , the set  $l_p - \bigcup_{q < p} l_q$  is even spaceable.

Let  $\mathbb{Z}_+$  be the set of non-negative integers equipped with discrete topology,  $M(\mathbb{Z}_+)$  be the space of all Radon measures on  $\mathbb{Z}_+$ , and  $p_0$  be a fixed prime number. For any  $k \in \mathbb{Z}_+$  and distinct non-zero  $m, n \in \mathbb{Z}_+$  we put

$$\delta_k * \delta_0 = \delta_0 * \delta_k := \delta_k,$$

$$\delta_n * \delta_n := \frac{1}{p_0^{n-1}(p_0 - 1)} \delta_0 + \sum_{k=1}^{n-1} p_0^{k-1} \delta_k + \frac{p_0 - 2}{p_0 - 1} \delta_n,$$

and  $\delta_m * \delta_n := \delta_{\max\{m,n\}}$ . Then, by [5],  $\mathbb{Z}_+$  equipped with the convolution  $*$  :  $M(\mathbb{Z}_+) \times M(\mathbb{Z}_+) \rightarrow M(\mathbb{Z}_+)$  defined by

$$\mu * \nu := \int_{\mathbb{Z}_+} \int_{\mathbb{Z}_+} \delta_m * \delta_n d\mu(m) d\nu(n), \quad (\mu, \nu \in M(\mathbb{Z}_+),$$

and the identity mapping on  $\mathbb{Z}_+$  as involution, is a commutative discrete hypergroup. Also, the measure  $m$  defined by

$$m(\{k\}) := \begin{cases} 1, & \text{if } k = 0, \\ (p_0 - 1)p_0^{k-1}, & \text{if } k \geq 1, \end{cases}$$

is a Haar measure for  $\mathbb{Z}_+$ . For more details about locally compact hypergroups see [2].

In the sequel, we consider  $\mathbb{Z}_+$  with above structure, and for each  $s > 0$  we denote  $L^s(\mathbb{Z}_+) := L^s(\mathbb{Z}_+, m)$ .

In this paper, by an elementary technique, we give an algorithm that generates lots of sequences in  $L^p(\mathbb{Z}_+)$  but not in  $L^q(\mathbb{Z}_+)$  for every  $q < p$ . This argument can be regarded as a variant of the technique that was used to prove the Banach-Steinhaus Theorem in [8]. Also, we prove that for each  $r > 1$ , the set  $L^r(\mathbb{Z}_+) - \bigcup_{q < r} L^q(\mathbb{Z}_+)$  is spaceable in  $L^r(\mathbb{Z}_+)$ .

## 2 Main Results

This is well-known that:

**Lemma 1.** *There is a sequence  $(a_n)_{n=1}^\infty$  of real numbers such that  $a_n > 1$  for all  $n$ ,  $\lim_{n \rightarrow \infty} a_n = 1$ , and  $\sum_{n=1}^\infty \frac{1}{n^{a_n}}$  converges.*

**Lemma 2.** *If  $0 < q < r < \infty$ , then  $L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+)$ .*

*Proof.* Let  $f := (x_n)_{n=0}^\infty \in L^q(\mathbb{Z}_+)$ . Since

$$\|f\|_q^q = |x_0|^q + \sum_{n=1}^\infty |x_n|^q (p_0 - 1)p_0^{n-1} < \infty, \quad (2.1)$$

there is a positive number  $N$  such that for each  $n \geq N$ ,

$$|x_n| ((p_0 - 1)p_0^{n-1})^{\frac{1}{q}} < 1.$$

But for each  $n \in \mathbb{N}$ ,

$$((p_0 - 1)p_0^{n-1})^{\frac{1}{q}} \geq 1.$$

Hence, for all  $n \geq N$ ,  $|x_n| < 1$ , and then  $|x_n|^r < |x_n|^q$ . Therefore, for each  $n \geq N$  we have

$$|x_n|^r (p_0 - 1)p_0^{n-1} < |x_n|^q (p_0 - 1)p_0^{n-1}.$$

By (2.1) we get  $\sum_{n=1}^{\infty} |x_n|^r (p_0 - 1)p_0^{n-1} < \infty$ , i.e.  $f \in L^r(\mathbb{Z}_+)$ .  $\square$

**Lemma 3.** *Let  $q < r$ ,  $p_0$  be a prime number, and  $(a_n)_{n=1}^{\infty}$  be a sequence as in Lemma 1. If the sequence  $\mathbf{x} = (x_n)_{n=0}^{\infty}$  is defined by*

$$x_n := \begin{cases} 1 & \text{if } n = 0, \\ \frac{1}{(p_0^{n-1}(p_0-1)n^{a_n})^{\frac{1}{r}}} & \text{if } n \geq 1, \end{cases}$$

then  $\mathbf{x} \in L^r(\mathbb{Z}_+) - L^q(\mathbb{Z}_+)$ .

*Proof.* From

$$\begin{aligned} \|\mathbf{x}\|_r^r &= \sum_{n=0}^{\infty} |x_n|^r m(\{n\}) \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{p_0^{n-1}(p_0-1)n^{a_n}} (p_0-1)p_0^{n-1} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n^{a_n}} < \infty \end{aligned}$$

we conclude that  $\mathbf{x} \in L^r(\mathbb{Z}_+)$ . Let  $0 < q < r$ . Since  $\frac{r}{a_n} < r$  for every  $n$  and  $\lim_{n \rightarrow \infty} \frac{r}{a_n} = r$ , there is some  $t_0 \in \mathbb{N}$  such that  $q < \frac{r}{a_{t_0}}$ . Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{t_0}} = \frac{1}{a_{t_0}} < 1,$$

we can choose  $s \in (\frac{1}{a_{t_0}}, 1)$ , and there is  $N_0 \in \mathbb{N}$  such that  $\frac{a_n}{a_{t_0}} < s < 1$  for all  $n \geq N_0$ . Hence,

$$\frac{1}{n^s} < \frac{1}{n^{\frac{a_n}{a_{t_0}}}}$$

for all  $n \geq N_0$ . We have

$$\begin{aligned} \|\mathbf{x}\|_{\frac{r}{a_{t_0}}}^{\frac{r}{a_{t_0}}} &= \sum_{n=0}^{\infty} |x_n|^{\frac{r}{a_{t_0}}} m(\{n\}) \\ &= 1 + \sum_{n=1}^{\infty} \left( \frac{1}{(p_0^{n-1}(p_0-1)n^{a_n})^{\frac{1}{r}}} \right)^{\frac{r}{a_{t_0}}} (p_0-1)p_0^{n-1} \\ &= 1 + \sum_{n=1}^{\infty} ((p_0-1)p_0^{n-1})^{1-\frac{1}{a_{t_0}}} \frac{1}{n^{\frac{a_n}{a_{t_0}}}}. \end{aligned}$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n^s} = \infty$  and

$$((p_0-1)p_0^{n-1})^{1-\frac{1}{a_{t_0}}} \geq 1,$$

we have

$$\sum_{n=N_0}^{\infty} \frac{1}{n^s} \leq \sum_{n=N_0}^{\infty} \frac{1}{n^{\frac{a_n}{t_0}}} \leq 1 + \sum_{n=1}^{\infty} ((p_0-1)p_0^{n-1})^{1-\frac{1}{a_{t_0}}} \frac{1}{n^{\frac{a_n}{a_{t_0}}}} \leq \|\mathbf{x}\|_{\frac{r}{a_{t_0}}}^{\frac{r}{a_{t_0}}},$$

and so,  $\mathbf{x} \notin L^{\frac{r}{a_{t_0}}}(\mathbb{Z}_+)$ . Therefore,  $\mathbf{x} \notin L^q(\mathbb{Z}_+)$  because  $q < \frac{r}{a_{t_0}}$  and  $L^q(\mathbb{Z}_+) \subseteq L^{\frac{r}{a_{t_0}}}(\mathbb{Z}_+)$ .  $\square$  QED

**Theorem 1.** *If  $1 \leq q < r < \infty$ , then  $L^q(\mathbb{Z}_+)$  is not closed in  $L^r(\mathbb{Z}_+)$ .*

*Proof.* First Proof: Let  $C_c(\mathbb{Z}_+)$  be the space of all functions from  $\mathbb{Z}_+$  into  $\mathbb{C}$  with finite support. We have  $C_c(\mathbb{Z}_+) \subseteq L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+)$ , and  $C_c(\mathbb{Z}_+)$  is dense in  $L^q(\mathbb{Z}_+)$  and in  $L^r(\mathbb{Z}_+)$ . So, if  $L^q(\mathbb{Z}_+)$  is closed in  $L^r(\mathbb{Z}_+)$ , then we have  $L^q(\mathbb{Z}_+) = L^r(\mathbb{Z}_+)$ , a contradiction.

Second Proof: Let  $T : L^q(\mathbb{Z}_+) \rightarrow L^r(\mathbb{Z}_+)$  be the inclusion mapping. We claim that  $T$  is continuous. For this, suppose that  $(f_n)_{n=0}^{\infty}$  is a sequence in  $L^q(\mathbb{Z}_+)$  with  $f_n := (x_{n,i})_{i=0}^{\infty}$ , and  $f_n \rightarrow f$  in  $L^q(\mathbb{Z}_+)$ , where  $f := (x_i)_{i=0}^{\infty} \in L^q(\mathbb{Z}_+)$ . For each  $0 < \varepsilon < 1$  there is  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$\|f_n - f\|_q^q = |x_{n,0} - x_0|^q + \sum_{i=1}^{\infty} |x_{n,i} - x_i|^q (p_0 - 1)p_0^{i-1} < \varepsilon < 1.$$

Hence,  $|x_{n,i} - x_i|^q < 1$  for all  $n \geq N$  and  $i \in \mathbb{Z}_+$ . Since  $q < r$ , we get

$$|x_{n,i} - x_i|^r \leq |x_{n,i} - x_i|^q,$$

where  $n \geq N$  and  $i \in \mathbb{Z}_+$ , and so  $\|f_n - f\|_r \leq \|f_n - f\|_q$  for all  $n \geq N$ . Therefore,  $f_n \rightarrow f$  with respect to the norm  $\|\cdot\|_r$ , and  $T$  is continuous.

Now, we show that  $\|\cdot\|_q$  and  $\|\cdot\|_r$  are not equivalent. In contrast, suppose that there exists  $M > 0$  such that for each  $f \in L^q(\mathbb{Z}_+)$ ,  $\|f\|_q \leq M\|f\|_r$ . For any  $n \in \mathbb{N}$ , let  $\chi_n$  be the characteristic function of the set  $\{n\}$ . Then for all  $n \in \mathbb{N}$ , we have

$$((p_0 - 1)p_0^{n-1})^{\frac{1}{q}} = \|\chi_n\|_q \leq M \|\chi_n\|_r = M ((p_0 - 1)p_0^{n-1})^{\frac{1}{r}},$$

and so,

$$((p_0 - 1)p_0^{n-1})^{\frac{1}{q} - \frac{1}{r}} \leq M,$$

which is a contradiction, since  $\frac{1}{q} - \frac{1}{r} > 0$ . Therefore, the norms  $\|\cdot\|_q$  and  $\|\cdot\|_r$  are not equivalent. Finally, suppose that  $L^q(\mathbb{Z}_+)$  is closed in  $L^r(\mathbb{Z}_+)$ . Since  $T$  is continuous, the induced mapping

$$\tilde{T} : L^q(\mathbb{Z}_+) \longrightarrow T(L^q(\mathbb{Z}_+)) = (L^q(\mathbb{Z}_+), \|\cdot\|_r), \quad \tilde{T}(f) := T(f) = f,$$

is a bijective continuous linear operator between Banach spaces. So, by the Open Mapping Theorem,  $\tilde{T}$  is an isomorphism, and in particular the norms  $\|\cdot\|_q$  and  $\|\cdot\|_r$  are equivalent on  $L^q(\mathbb{Z}_+)$ , a contradiction. Therefore,  $L^q(\mathbb{Z}_+)$  is not closed in  $L^r(\mathbb{Z}_+)$ .  $\square$

By [9, Theorem 2.4], we can conclude:

**Corollary 1.** *If  $0 < q < r < \infty$ , then  $L^r(\mathbb{Z}_+) - L^q(\mathbb{Z}_+)$  is spaceable.*

Here, we recall the following result (see [9, Theorem 3.3]).

**Theorem 2.** *Suppose that  $X$  is a Frechet space, and for each  $n = 1, 2, \dots$ ,  $Z_n$  is a Banach space. Let for each  $n \in \mathbb{N}$ ,  $T_n : Z_n \longrightarrow X$  be bounded linear operators. If  $Y := \text{span}(\bigcup_{n=1}^{\infty} T_n(Z_n))$  is not closed in  $X$ , then  $X - Y$  is spaceable.*

**Theorem 3.** *If  $r > 1$ , then*

$$L^r(\mathbb{Z}_+) - \bigcup_{q < r} L^q(\mathbb{Z}_+)$$

*is spaceable.*

*Proof.* The sequence  $\mathbf{x} \in L^r(\mathbb{Z}_+)$  constructed in Lemma 3 does not belong to  $L^q(\mathbb{Z}_+)$  for all  $q < r$ , i.e.  $L^r(\mathbb{Z}_+) - \bigcup_{q < r} L^q(\mathbb{Z}_+)$  is non-empty.

Let  $q < r$ . Since  $C_c(\mathbb{Z}_+)$  is dense in  $L^r(\mathbb{Z}_+)$ , and

$$C_c(\mathbb{Z}_+) \subseteq L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+),$$

$L^q(\mathbb{Z}_+)$  is dense in  $L^r(\mathbb{Z}_+)$ , and so  $\bigcup_{q < r} L^q(\mathbb{Z}_+)$  is not closed in  $L^r(\mathbb{Z}_+)$ . By Lemma 2, the set  $\bigcup_{q < r} L^q(\mathbb{Z}_+)$  is actually a countable union:

$$\bigcup_{q < r} L^q(\mathbb{Z}_+) = \bigcup_{n=1}^{\infty} L^{q_n}(\mathbb{Z}_+), \quad (2.2)$$

where  $q_1 < q_2 < \dots < r$ , and moreover, the sequence  $(q_n)_n$  converges to  $r$ . Easily, one can see that  $\text{span}(\bigcup_{n=1}^{\infty} L^{q_n}(\mathbb{Z}_+)) = \bigcup_{n=1}^{\infty} L^{q_n}(\mathbb{Z}_+)$ . So, applying Theorem 2 (put  $Z_n := (L^{q_n}(\mathbb{Z}_+), \|\cdot\|_{q_n})$  and  $T_n : Z_n \hookrightarrow L^r(\mathbb{Z}_+)$ ),  $L^r(\mathbb{Z}_+) - \bigcup_{q < r} L^q(\mathbb{Z}_+)$  is spaceable.  $\square$

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